The Discretized Coulomb Friction Model in a Non-Singular Complementarity Formulation for Multibody Systems with Contacts

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Summary. The dynamics of a multibody system with contacts and friction can be formulated as a linear complementarity problem (LCP) via the discretization of the friction cone. Several discrete friction models are available in the literature, and all of them produce rank-deficient LCPs due to the use of multiple directions to discretize the tangent plane. In this work, a new discrete friction model is proposed which couples the equations in these directions and allows the dynamic formulation to take the form of a full-rank LCP.

Abstract

The dynamics of multibody systems with contacts presents some well-known challenges, especially when it involves frictional contacts. If friction is neglected, the representation takes the form of a *linear complementarity problem* (LCP), for which the existance of solution is guaranteed, and several direct or iterative solution algorithms are available in the literature. However, considering friction in contacts using the Coulomb model turns the formulation into a *nonlinear complementarity problem* (NCP), for which the existence of solution is not guaranteed.

Several friction models have been proposed to approximate the Coulomb model so that the model can take the form of an LCP [1, 2, 3]. These models are based on the discretization of the friction cone, which turns the non-linear inequalities of the original model into linear ones by choosing a finite number of directions in the tangent plane. As a result, this discretized model approximates the friction cone with a pyramid. It is worth mentioning that this issue is only related to the contact points that are not sliding, because the static friction force is confined to the friction cone; whereas the kinetic friction force is defined by a constitutive relation.

Generally, there are two ways to discretize the Coulomb friction model. One approach is based on a *discrete force*, in which the directions are used to define non-negative components of the friction force $\boldsymbol{\lambda}_\mathrm{t} = \sum \beta_j \mathbf{e}_j$, being $\mathbf{e}_j$ the unit vectors associated with each direction of the tangent plane [2, 3]. The second approach is based on a *discrete acceleration*, in which the directions are used to define non-negative components of the pre-sliding tangential acceleration $\dot{\mathbf{u}}_\mathrm{t} = \sum \kappa_j \mathbf{e}_j$ [1]. The size of the problem to solve is very similar in both cases, and they can behave the same for certain sets of chosen directions.

A common characteristic of these models in multibody system formulations is that the lead matrix of the LCP is rank deficient, even though there is no redundancy in the contact forces of the system. This is due to the use of more than two non-idependent directions in the tangent plane, and so it is an artifact of the model. Nevertheless, it is possible to define a full-rank LCP formulation using the discretized friction model proposed here.

Let us define k different directions in the tangent plane, so that the relative tangent acceleration can be discretized and parameterized as follows

$$\dot{\mathbf{u}}_\mathrm{t} = \sum_{j=1}^{k} \left(\kappa_{j+}\mathbf{e}_j - \kappa_{j-}\mathbf{e}_j\right) \tag{1}$$

where $\mathbf{e}_j$ are the unit vectors of these directions, and $\kappa_{j+} \geqslant 0$ and $\kappa_{j-} \geqslant 0$ are the positive acceleration parameters pointing to the positive and negative directions of $\mathbf{e}_j$, respectively. On the other hand, two *friction saturations* are defined in each direction to enforce the limits of the friction force in the positive and negative directions

$$\sigma_{j+} = \mu\lambda_\mathrm{n} - \mathbf{e}_j^\mathrm{T}\boldsymbol{\lambda}_\mathrm{t} + \rho\kappa_{j+} \geqslant 0 \tag{2}$$

$$\sigma_{j-} = \mu\lambda_\mathrm{n} + \mathbf{e}_j^\mathrm{T}\boldsymbol{\lambda}_\mathrm{t} + \rho\kappa_{j-} \geqslant 0 \tag{3}$$

where μ is the static friction coefficient, λ_n is the normal force, and ρ is the *phantom inertia*. The friction saturations σ need to be complementary to the parameters κ, so that sliding only starts when the friction force reaches the bounds at the opposite direction. This is enforced via the complementarity conditions $\sigma_{j+}\kappa_{j-} = 0$ and $\sigma_{j-}\kappa_{j+} = 0$.

The novelty of this model is the addition of the *phantom inertia* ρ into the equations, which does not change the dynamics of the contact model, but it adds some extra information into the equations. This makes the lead matrix of the LCP full rank, in spite of using a discrete friction model. The complementarity conditions make sure that force and acceleration will never point to the same direction, that is, adding that term does not change this. It can be shown that a solution where both parameters κ_{j+} and κ_{j-} are positive is not possible, no matter how big ρ is. When the friction force reaches a bound, only the acceleration pointing to the opposite direction will be positive; and when the friction force is within the bounds of that direction, i.e., $|\mathbf{e}_j^\mathrm{T}\boldsymbol{\lambda}_\mathrm{t}| < \mu\lambda_\mathrm{n}$, the two parameters $\kappa_{j+} = \kappa_{j-} = 0$. The inertia ρ only makes the friction saturation of one bound larger, in case the sliding starts in the oposite direction, and thus it does not influence the stick-slip transition, see Figure 1.

This model can be introduced in a formulation for multibody systems, where the dynamic representation takes the form of an LCP. Let $\mathbf{q}$ be the minimal set of n generalized coordinates of a multibody system, so that any bilateral constraint is

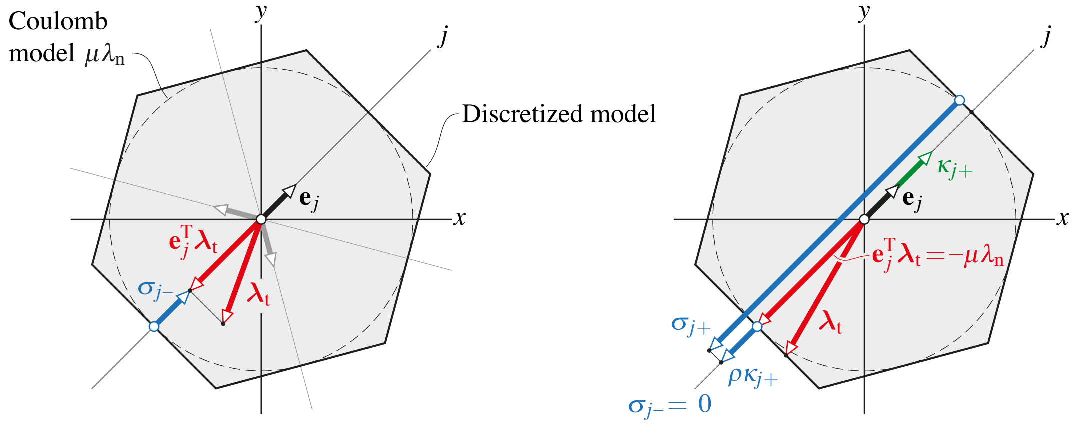

Figure 1: Discretized friction model, and model variables for the j-th direction. Static friction (left), and stick-slip transition (right).

embedded. A minimal set of n generalized velocities $\mathbf{v}$ is chosen, such that $\dot{\mathbf{q}} = \boldsymbol{\Gamma}\mathbf{v}$, where $\boldsymbol{\Gamma}(\mathbf{q},t)$ is the transformation matrix. With this representation, the dynamic equations can be written as

$$\mathbf{M}\dot{\mathbf{v}} + \mathbf{c} = \mathbf{f}_a + \mathbf{f}_t + \mathbf{f}_n \tag{4}$$

where $\mathbf{M}(\mathbf{q})$ is the $n\times n$ mass matrix, $\mathbf{c}(\mathbf{q},\mathbf{v})$ is the $n\times 1$ array containing the Coriolis and centrifugal terms, $\mathbf{f}_a$ is the generalized applied force, and $\mathbf{f}_t$ and $\mathbf{f}_n$ are the generalized tangential and normal contact forces, respectively.
The *gap functions* that define the distance between the r contacting surface pairs are arranged into the array $\boldsymbol{\Phi}(\mathbf{q}) \geq \mathbf{0}$. Therefore, the normal acceleration is $\ddot{\boldsymbol{\Phi}} = \dot{\mathbf{u}}_n = \mathbf{J}_n\dot{\mathbf{v}} + \dot{\mathbf{J}}_n\mathbf{v} \geqslant \mathbf{0}$, where $\mathbf{J}_n(\mathbf{q})$ is the $r\times n$ contact Jacobian matrix. Therefore, the generalized normal contact forces are $\mathbf{f}_n = \mathbf{J}_n^T\boldsymbol{\lambda}_n$, where the r normal contact forces are

$$\boldsymbol{\lambda}_n \geqslant \mathbf{0} \quad \text{complementary to} \quad \dot{\mathbf{u}}_n \geqslant \mathbf{0} \tag{5}$$

The tangent plane of each contact point is characterized by two orthogonal directions, in which the components of the sliding acceleration can be expressed as $\dot{\mathbf{u}}_t = \mathbf{J}_t\dot{\mathbf{v}} + \dot{\mathbf{J}}_t\mathbf{v}$, where $\mathbf{J}_t(\mathbf{q})$ is the $2r\times n$ friction Jacobian matrix. The generalized friction force is then $\mathbf{f}_t = \mathbf{J}_t^T\boldsymbol{\lambda}_t$, where $\boldsymbol{\lambda}_t$ contains the $2r$ friction contact force components. The discrete friction model requires a set of directions in the tangent plane, so that the components of their k_i unit vectors of the i-th contact point can be arranged as columns of the matrix $\mathbf{E}_i = \left[\, \mathbf{e}_1 \ \cdots \ \mathbf{e}_{k_i} \,\right]$, and therefore, $\dot{\mathbf{u}}_{t_i} = \mathbf{E}_i\boldsymbol{\kappa}_{i+} - \mathbf{E}_i\boldsymbol{\kappa}_{i-}$. Equations (1), (2) and (3) can be arranged in a matrix form for each contact, so that the 2 arrays of k_i friction saturations are

$$\boldsymbol{\sigma}_{i+} = \boldsymbol{\mu}\lambda_{n_i} - \mathbf{E}_i^T\boldsymbol{\lambda}_{t_i} + \boldsymbol{\rho}_i\boldsymbol{\kappa}_{i+} \geqslant \mathbf{0} \quad \text{complementary to} \quad \boldsymbol{\kappa}_{i-} \geqslant \mathbf{0} \tag{6}$$

$$\boldsymbol{\sigma}_{i-} = \boldsymbol{\mu}\lambda_{n_i} + \mathbf{E}_i^T\boldsymbol{\lambda}_{t_i} + \boldsymbol{\rho}_i\boldsymbol{\kappa}_{i-} \geqslant \mathbf{0} \quad \text{complementary to} \quad \boldsymbol{\kappa}_{i+} \geqslant \mathbf{0} \tag{7}$$

where $\boldsymbol{\mu}_i = [\, \mu_i \ \cdots \ \mu_i \,]^T$, and $\boldsymbol{\rho}_i = \mathrm{diag}\,(\rho_i, \ldots, \rho_i)$. Then, the system dynamic equations can be writen as

$$\begin{bmatrix} \mathbf{M} & -\mathbf{J}_t^T & -\mathbf{J}_n^T & \mathbf{0} & \mathbf{0} \\ \mathbf{J}_t & \mathbf{0} & \mathbf{0} & -\mathbf{E} & \mathbf{E} \\ \mathbf{J}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^T & \boldsymbol{\mu} & \mathbf{0} & \boldsymbol{\rho} \\ \mathbf{0} & -\mathbf{E}^T & \boldsymbol{\mu} & \boldsymbol{\rho} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \boldsymbol{\lambda}_t \\ \boldsymbol{\lambda}_n \\ \boldsymbol{\kappa}_+ \\ \boldsymbol{\kappa}_- \end{bmatrix} + \begin{bmatrix} \mathbf{c} - \mathbf{f}_a \\ \dot{\mathbf{J}}_t\mathbf{v} \\ \dot{\mathbf{J}}_n\mathbf{v} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \dot{\mathbf{u}}_n \\ \boldsymbol{\sigma}_- \\ \boldsymbol{\sigma}_+ \end{bmatrix} \tag{8}$$

where $\mathbf{E} = \mathrm{diag}\,(\mathbf{E}_1, \ldots, \mathbf{E}_r)$, $\boldsymbol{\mu} = \mathrm{diag}\,(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r)$, and $\boldsymbol{\rho} = \mathrm{diag}\,(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_r)$. The variables $\dot{\mathbf{v}}$ and $\boldsymbol{\lambda}_t$, which do not have any complementarity condition, can be eliminated from the system via the Schur complement, so that an LCP with the variables $\boldsymbol{\lambda}_n$, $\boldsymbol{\kappa}_+$ and $\boldsymbol{\kappa}_-$, and the complementarity conditions in Eqn. (5), (6) and (7) can be formulated.
The role of the square diagonal *phantom inertia* matrix $\boldsymbol{\rho}$ is clearly shown in equation (8). Due to the use of dependent directions to discretize the friction cone, the two last block columns of the lead matrix would be linearly dependent. However, the discrete friction model proposed here gives rise to a full-rank LCP formulation, by coupling the friction saturations without changing the dynamics of the system. If the contact forces are not redundant, the contact normal and friction Jacobian matrices are full-rank, and so is the lead matrix. Nevertheless, if the system presents redundancy, some physics-based techniques (e.g., constraint relaxation, minimum-norm solution, etc.) would be required to solve the dynamic equations and determine the contact forces.

References

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