ECCOMAS Thematic Conference on Multibody Dynamics
June 19-22, 2017, Prague, Czech Republic

On the Generalized Friction Cone for Multibody Systems

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Abstract

The use of the Coulomb friction model is considered to be representative for modelling contact. One of the most important element in this model is the *friction cone*. It arises from the fact that the static friction force has a threshold value, and therefore, all the possible contact force vectors of a sticking contact point must lie whithin the cone.

The *generalized friction cone* [1, 2] takes the dynamics of the system into account, and interprets the friction cone in the configuration space of the system. This representation is very useful to analize different phenomena related to friction, such as the *Painlevé paradox* [2]. However, the fact that this cone is in the multi-dimentional configuration space makes it hard to visualize it. Here, the equation of the generalized friction cone projected to the contact velocity space is derived, so that it can be used to represent and visualize it in a 3-dimentional space.

Assuming isotropy in the tangent plane of the contact point, the limit of the friction force is defined as

$$\|\boldsymbol{\lambda}_\mathrm{t}\| = \sqrt{\boldsymbol{\lambda}_\mathrm{t}^\mathrm{T}\boldsymbol{\lambda}_\mathrm{t}} \leqslant \mu\lambda_\mathrm{n} \tag{1}$$

where $\boldsymbol{\lambda}_\mathrm{t}$ is the friction force, λ_n is the normal force, and μ is the friction coeficient, which is assumed to be equal for the static and kinetic friction. Equation (1) represents the *classic friction cone* (κ_μ), with the quadratic matrix form

$$\kappa_\mu(\boldsymbol{\lambda}_\mathrm{c}) = \boldsymbol{\lambda}_\mathrm{c}^\mathrm{T}\mathbf{Q}_\mu\boldsymbol{\lambda}_\mathrm{c} = \begin{bmatrix}\boldsymbol{\lambda}_\mathrm{t}\\ \lambda_\mathrm{n}\end{bmatrix}^\mathrm{T}\begin{bmatrix}\mathbf{I}_{2\times 2} & \mathbf{0}\\ \mathbf{0} & -\mu^2\end{bmatrix}\begin{bmatrix}\boldsymbol{\lambda}_\mathrm{t}\\ \lambda_\mathrm{n}\end{bmatrix} \leqslant 0 \tag{2}$$

where $\lambda_\mathrm{n} \geqslant 0$, and $\mathbf{I}_{2\times 2}$ is the 2×2 identity matrix.

For the analysis of the contact dynamics in a multibody system, it is useful to consider a *reduced representation* of the system in the space of the contact velocities

$$\mathbf{u}_\mathrm{c} = \begin{bmatrix}\mathbf{u}_\mathrm{t}\\ u_\mathrm{n}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_\mathrm{t}\mathbf{v}\\ \mathbf{A}_\mathrm{n}\mathbf{v}\end{bmatrix} = \mathbf{A}\mathbf{v} \tag{3}$$

where $\mathbf{u}_\mathrm{t}$ and u_n are the tangential and normal relative velocity components at the contact point of interest, $\mathbf{A}$ is the contact Jacobian matrix, and $\mathbf{v}$ contains the generalized velocities of the system with the mass matrix $\mathbf{M}$. The infinitesimal change of the contact velocities can be related to the contact forces by the effective mass matrix $\mathbf{M}_\mathrm{c} = \left(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^\mathrm{T}\right)^{-1}$, so that the differential of the contact force impulse $\mathrm{d}\Lambda_\mathrm{c} = \boldsymbol{\lambda}_\mathrm{c}\mathrm{d}t = \mathbf{M}_\mathrm{c}\delta\mathbf{u}_\mathrm{c}$. Note that the incremental change $\delta\mathbf{u}_\mathrm{c}$ only accounts for the contact forces. Nevertheless, other forces might also contribute to the total incremental change of the contact velocities $\mathrm{d}\mathbf{u}_\mathrm{c} = \delta\mathbf{u}_\mathrm{c} + \delta\mathbf{u}_\mathrm{c}^0$, where $\delta\mathbf{u}_\mathrm{c}^0$ accounts for the change of $\mathbf{u}_\mathrm{c}$ due to the rest of forces acting on the system.

A quadratic expression for the contact velocity changes $\delta\mathbf{u}_\mathrm{c}$ can be derived from Eqn. (2) by using the aforementioned expression,

$$\kappa_\mathrm{g}(\delta\mathbf{u}_\mathrm{c}) = \delta\mathbf{u}_\mathrm{c}^\mathrm{T}\mathbf{Q}\delta\mathbf{u}_\mathrm{c} = \delta\mathbf{u}_\mathrm{c}^\mathrm{T}\left(\mathbf{M}_\mathrm{c}\mathbf{Q}_\mu\mathbf{M}_\mathrm{c}\right)\delta\mathbf{u}_\mathrm{c} = \begin{bmatrix}\delta\mathbf{u}_\mathrm{t}\\ \delta u_\mathrm{n}\end{bmatrix}^\mathrm{T}\begin{bmatrix}\mathbf{Q}_\mathrm{t} & \mathbf{Q}_\mathrm{tn}\\ \mathbf{Q}_\mathrm{tn}^\mathrm{T} & Q_\mathrm{n}\end{bmatrix}\begin{bmatrix}\delta\mathbf{u}_\mathrm{t}\\ \delta u_\mathrm{n}\end{bmatrix} \leqslant 0 \tag{4}$$

where $\mathbf{Q}_\mathrm{t}$ and Q_n are characteristic elements of the matrix $\mathbf{Q}$ and will be defined below. This homogenious quadratic equation represents the projection of the generalized cone into the contact velocity space.

As in the clasic cone κ_μ, the friction coefficient affects the geometry of the generalized cone κ_g. For $\mu = 0$ the cone degenerates into a line given by the parametrization $\delta\mathbf{u}_\mathrm{c} = \mathbf{A}\mathbf{M}^{-1}\mathbf{A}_\mathrm{n}\lambda_\mathrm{n}\mathrm{d}t$. This line represents the *space of constrained motion* associated with the contact constraint projected into the contact velocity space, also known as *natural contact direction* in [3]. It can be interpreted as the direction in which the contact velocity varies due to the normal contact force alone. In case of frictionless collisions, this direction is important because all non-impusive forces are usually neglected and only the impulses of the normal force are taken into account. On the other hand, for $\mu \to \infty$ the cone degenerates into a plane given by the parametrization $\delta\mathbf{u}_\mathrm{c} = \mathbf{A}\mathbf{M}^{-1}\mathbf{A}_\mathrm{t}\boldsymbol{\lambda}_\mathrm{t}\mathrm{d}t$. This plane is not directly related to either the natural contact direction or the plane $\delta u_\mathrm{n} = 0$, see Figure 1.

In general, $\mathbf{Q}$ is a full-rank symmetric matrix that represents an elliptic cone κ_d without any particular shape. Nevertheless, its geometry in some cases can be of interest and help to better understand the dynamics of multibody systems with frictional contacts. For instance, it is clear from Eqn. (4) that the direction $\delta\mathbf{u}_\mathrm{t} = \mathbf{0}$ is located inside

the cone if and only if $Q_n \leqslant 0$. It can be shown that this occurs for values of the friction coefficient greater than a critical value [4], $\mu \geqslant \mu_{crit} = \|\mathbf{M}_t\mathbf{h}\| = \sqrt{\mathbf{h}^T\mathbf{M}_t^2\mathbf{h}}$, where $\mathbf{M}_t = (\mathbf{A}_t\mathbf{M}^{-1}\mathbf{A}_t^T)^{-1}$ and $\mathbf{h} = \mathbf{A}_t\mathbf{M}^{-1}\mathbf{A}_n^T$ depend on the configuration of the system. The *critical friction coefficient* μ_{crit} plays an important role in single-point collisions with friction, in which sliding cannot restart if $\mu \geqslant \mu_{crit}$. This fact is consistent with the dynamic cone, because $\delta\mathbf{u}_t = \mathbf{0}$ is possible in such a case (i.e., it is inside the cone, see Figure 1), and therefore, the contact force alone can keep the contact point without sliding.

Another particular aspect of the generalized cone is its intersection with the plane $\delta u_n = 0$, which is a degenerate conic described by the quadratic equation $\delta\mathbf{u}_t^T\mathbf{Q}_t\delta\mathbf{u}_t = 0$. The intersection is different from a point (the vertex of the cone) if and only if $\det \mathbf{Q}_t \leqslant 0$, and it can be shown that this occurs if the friction coefficient is greater than a value, $\mu \geqslant \mu_{jam} = \|M_n\mathbf{h}\|^{-1} = \left(M_n\sqrt{\mathbf{h}^T\mathbf{h}}\right)^{-1}$, where $M_n = (\mathbf{A}_n\mathbf{M}^{-1}\mathbf{A}_n^T)^{-1}$ also depends on the configuration of the system. In such a case, *dynamic jamming* (or *jamb*) [1, 2, 4] can happen if the contact point is sliding in a particular direction which gives $\delta u_n \leqslant 0$. This phenomenon gave rise to the *Painlevé paradox*, in which the dynamic equations of a rigid body with Coulomb friction in the contacts are shown to have no solution for certain kinematic states (i.e., configuration and velocity), or even several possible solutions [1, 2].

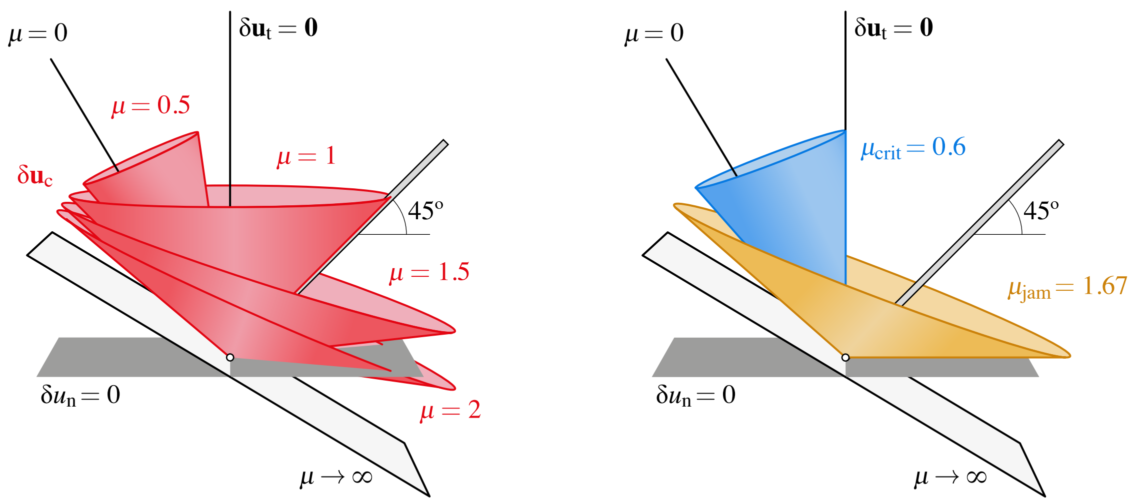

Figure 1: Projection of the generalized friction cone (κ_g) to the contact velocity space of a rod at 45° with the ground for different friction coefficients.

As an example, let us consider a single rod in contact with the ground at 45°. Figure 1 shows the generalized cone for different friction coefficients, as well as the limit cases $\mu = 0$ and $\mu \to \infty$. All the particular cases discussed above are also shown, and even though it is just one body in contact, this example is representative of a general case. Moreover, it is also shown the cone for the caracteristic varlues μ_{crit} and μ_{jam}. For high friction coefficient ($\mu > \mu_{jam}$), it can be seen how sliding to certain directions gives $\delta u_n < 0$, which presents the paradoxal situation where the dynamic equations have several solutions or none.

This new representation of the generalized friction cone gives a geometric tool that helps to understand the dynamics of multibody systmes with frictional contacts. It is not only consistent with all the theories involving the Coulomb friction model [1, 2, 4], but it also captures the paradoxal behaviour of the model.

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